High-Quality Approximation of Eigenvalues in Structural Optimization

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A new function for approximating natural frequency constraints during structural optimization is presented. Modern procedures for optimizing a structure typically solve a sequence of approximate subproblems as a means of efficiently finding the solution to the full-design problem. The subproblem is constructed by approximating the response quantities with a first-order Taylor series approximation to the actual functions, using appropriate intermediate design variables. The nonlinearity of frequency constraint functions has historically required the imposition of strict move limits or the use of a second-order Taylor series approximation. The Rayleigh Quotient Approximation (RQA) presented here increases the quality of the approximate frequency constraint by approximating the modal strain and kinetic energies instead of the frequency eigenvalue itself. Numerical examples demonstrate that the Rayleigh Quotient Approximation achieves fast and stable convergence with generous move limits.

Introduction

Structural Optimization Problem

THE purpose of this study is to examine a function for approximating natural frequency constraints during structural optimization. The nonlinearity of frequencies has posed a barrier to constructing approximations for frequency constraints of high enough quality to facilitate efficient solutions. A new function to represent frequency constraints, called the Rayleigh Quotient Approximation (RQA), is presented. Its ability to better represent the actual frequency constraint results in stable convergence with less restrictive move limits than a conventional Taylor Series Approximation (TSA) of the eigenvalue.

The objective of the optimization problem is to minimize structural weight subject to some minimum (or maximum) allowable frequency and perhaps subject to other constraints, such as stress, displacement, and gage size, as well. A reason for constraining natural frequencies during design might be to avoid potential resonant frequencies due to machinery or actuators connected to the structure. Another reason might be to satisfy requirements of an aircraft's or spacecraft's control law. Whatever the structure supports may be sensitive to a frequency band that must be avoided. Any of these situations, or others, may require the designer to insure the satisfaction of frequency constraints. A further motivation for considering accurate approximations of natural frequencies is that they are fundamental to dynamic response constraints. Techniques for natural frequency constraints may have application to transient response and frequency response problems.

Constraint Approximations

Engineers have long used the Taylor Series Approximation as a tool to simplify problems. In 1974, Schmit and Farshi exploited the use of TSA's to form approximate subproblems to the actual design problem.[1] Since then, much attention has been focused on finding the most appropriate intermediate design variables to use for the best TSA. Schmit and Miura originally championed the use of reciprocal variables.[2] Starnes and Haftka[3] and Fleury and Braibant[4] have shown that a hybrid constraint using mixed variables (i.e., a combination of direct and reciprocal variables) yields a more conservative approximation. Woo generalized the concept of mixed variables in his Generalized Hybrid Constraint (GHC) Approximation, where a variable exponent controls how conservative is the convex approximation.[5] Fleury devised a means of selecting an "optimal" intermediate variable based on second-order information.[6] Vanderplaats and Salajegheh demonstrated improved quality for constraint approximations in the element property space of frame elements when the optimization design variables are cross-sectional dimensions.[7]

All of these approaches sought improvement by selecting the "best" choice of intermediate variables. Few questioned whether a Taylor series to approximate the response quantity being constrained was the best way to generate the approximate problem. In 1985, Mills-Curran and Schmit used linear approximations of the eigenvalues and eigenvectors to formulate maximum displacement constraints on the dynamic response of damped structures.[8] In 1987, Vanderplaats and Salajegheh demonstrated that, for stress constraints, using a Taylor series to approximate the internal loads instead of the stresses themselves could increase the rate of convergence and reduce the need for move limits.[9] They observed that internal loads are a more fundamental quantity than stresses. Reliable stresses can be calculated once the internal loads are known. Generalizing this observation leads one to ask for frequency constraints: What might be a more fundamental quantity than its eigenvalue?

Frequency Constraints

By nature, frequency constraints exhibit greater nonlinearity than static constraints. The nonlinearity is readily observed through the appearance of the same cross-sectional variables in both the numerator and denominator of Rayleigh's quotient. Venkayya and Tischler have pointed out that, in practical structures, the denominator (kinetic energy) is typically dominated by the nonstructural mass.[10] In this case, frequency eigenvalues are more nearly linear in the cross-sectional property (direct design variable) space. Based on this assumption, some researchers have preferred a Taylor series constructed in the direct design variable space.[7] On the other hand, Miura and Schmit presented results that were better in the reciprocal design space than in the direct design space.[11] Nevertheless,

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their studies revealed that the eigenvalues are highly nonlinear in both direct and reciprocal design variable space, so that strict move limits are usually required. As a result, they proposed using a second-order instead of first-order Taylor series. Although the second-order approximation provided stable convergence without strict move limits, they reported the total computational time was "comparable with that required using first order approximations with move limits."[11]

Rayleigh Quotient Approximation

Venkayya's approach in formulating the optimality conditions for frequency constraints[10,12] suggests that modal strain energy and modal kinetic energy may be better quantities to approximate than the eigenvalue. The eigenvalue for a natural frequency is defined by the Rayleigh quotient, that is, the ratio of a structure's modal strain energy to its modal kinetic energy.[13] Each of these quantities has an explicit linear dependence on the direct design variables when the element stiffness and mass are linear in the element's cross-sectional properties. This observation suggests that a frequency constraint might be better approximated by a separate Taylor series for the numerator and denominator in the Rayleigh quotient. In fact, the concept is similar to an alternative approximation proposed by Fox and Kapoor,[14] except that here the eigenvector's first-order estimate is not used. Venkayya's successful technique for scaling frequency constraints to a feasible design lends credibility to assuming, at least for some problems, that the sensitivity of an eigenmode to changes in the cross-sectional properties is relatively small.[15] Hence, the sensitivity of the eigenmode is ignored when forming TSA's to modal strain and kinetic energies in the current approach.

Theory

Mathematical Statement of Problem

The structural optimization problem is stated mathematically as minimizing an objective function, the weight W

$$\min W(\boldsymbol{x}) \tag{1}$$

subject to constraints

$$g_j(\boldsymbol{x}) \leq 0, \qquad j = 1,\ldots,m \tag{2}$$

and side constraints

$$x_i^l \leq x_i \leq x_i^u, \qquad i = 1,\ldots,n \tag{3}$$

where $\boldsymbol{x}$ is a vector of n design variables, $\boldsymbol{x}^l$ and $\boldsymbol{x}^u$ represent their lower and upper bounds, respectively, and $\boldsymbol{g}$ is the m inequality constraints. The design variables are linked to one or more of the p physical variables, represented by the vector $\boldsymbol{d}$ through a transformation matrix $\boldsymbol{T}$.

$$\mathrm{d}_k = \sum_{i=1}^{n} \mathrm{T}_{ki}\mathrm{x}_i, \qquad k = 1,\ldots,p \tag{4}$$

In general, the $\boldsymbol{T}$ matrix may be fully populated; however, each row of $\boldsymbol{T}$ is limited to only one nonzero element (so-called *group linking*) when reciprocal variable approximations are considered. In this case, the summation in Eq. (4) is unnecessary. For the examples below, which use rod and membrane elements exclusively, the design variables are the cross-sectional properties: rod areas and membrane thicknesses.

Frequency constraints are formed using the eigenvalue λ (square of the angular frequency ω) normalized by its allowable value (square of the allowable angular frequency)

$$g = \pm\left(\frac{\lambda}{\lambda^{\text{allow}}} - 1\right) \tag{5}$$

The positive sign is used for an upper bound and the minus sign for a lower bound. Only lower bound allowables are given in the following examples, since minimizing structural weight drives frequencies toward zero. Other constraints can be cast in the form of Eq. (5) by replacing the λ's with the appropriate response quantity (e.g., Von Mises stress or displacement value).

An approximation to the actual optimization problem is constructed by estimating the constraints using a first-order Taylor series (conventional TSA).

$$\bar{g}_{Dj} = g_{oj} + \sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i}(x_i - x_{oi}) \tag{6}$$

If the approximate problem is solved in the reciprocal design variable space (i.e., $\beta = 1/x$), then the approximate constraint function, later referred to as a reciprocal TSA of the eigenvalue, is

$$\bar{g}_{Rj} = g_{oj} - \sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i}\left(\frac{1}{x_i} - \frac{1}{x_{oi}}\right)x_{oi}^2 \tag{7}$$

The hybrid constraint uses either a direct or reciprocal variable, depending on the sign of the constraint's derivative for each design variable. This creates a convex and more conservative approximation. As generalized by Woo, the equations for the GHC approximation are

$$\bar{g}_{\mathrm{GHC}j} = g_{oj} + \sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i}(x_i - x_{oi})\left(\frac{x_i}{x_{oi}}\right)^r \tag{8a}$$

$$r = \begin{cases} p, \text{ if } \dfrac{\partial g_j}{\partial x_i} \geq 0 \\ p - n, \text{ if } \dfrac{\partial g_j}{\partial x_i} < 0 \end{cases} \tag{8b}$$

where p is a real number and n is a positive integer. When $p = 0$ and $n = 1$, the GHC reduces to Starnes and Haftka's hybrid constraint, or equivalently, Fleury and Braibant's Method of Mixed Variables.

The approximate subproblem formed with Eqs. (6), (7), or (8) is solved by a nonlinear programming optimization algorithm. Appropriate move limits are employed to insure that the design remains in the vicinity of the point about which the Taylor series was made. The move limits are applied as side constraints [Eq. (3)], if they are more restrictive than the minimum and maximum gage constraints that are otherwise used. Move limits are typically specified as a percentage of the current design variables. Alternatively, a move limit factor f determines the upper and lower bounds.

$$\frac{x_i}{f} \leq x_i \leq f x_i, \qquad i = 1,\ldots,n \tag{9}$$

Rayleigh Quotient Approximation

The relationship of a natural frequency ω or corresponding eigenvalue λ to its associated eigenvector ϕ and the system's stiffness and mass is expressed by Rayleigh's quotient

$$\lambda = \omega^2 = \frac{\phi^t \boldsymbol{K} \phi}{\phi^t \boldsymbol{M} \phi} = \frac{U}{T} \tag{10}$$

where the modal strain energy U and the modal kinetic energy T are the sum of the strain and kinetic energies, respectively, from each of the elements. The structural system's mass and stiffness matrices can be represented as

$$\boldsymbol{K} = \boldsymbol{K}_o + \sum_{i=1}^{n} \boldsymbol{K}_i' x_i \tag{11a}$$

$$M = M_o + \sum_{i=1}^{n} M_i' x_i \tag{11b}$$

where K_i' and M_i' are the sensitivity of the stiffness and mass, respectively, to all the elements controlled by the ith design variable. For rod and membrane elements, the element stiffness and mass matrices are linear in the design variables, so that Eqs. (11) are exact. For frames the element matrices are functions of several dependent cross-sectional properties. If cross-sectional dimensions are used as design variables instead, Eqs. (11) are approximate. As Vanderplaats and Salajegheh point out, the cross-sectional properties are appropriate intermediate design properties for the constraint approximation even when designing for the cross-sectional dimensions directly.[7] The RQA below is entirely compatible with their approach of constructing constraint approximations in the cross-sectional property space.

Modal strain energy can be expressed as

$$U = u_o + \sum_{i=1}^{n} u_i' x_i \tag{12}$$

and modal kinetic energy is

$$T = t_o + \sum_{i=1}^{n} t_i' x_i \tag{13}$$

where

$$u_i' = \phi^t K_i' \phi \tag{14a}$$

$$t_i' = \phi^t M_i' \phi \tag{14b}$$

Here u_o is strain energy from undesigned elements, and t_o is the kinetic energy due to nonstructural mass and undesigned elements. The gradient of an eigenvalue, used in Eqs. (6), (7), or (8), is

$$\frac{\partial \lambda}{\partial x_i} = \frac{u_i' - \lambda t_i'}{T} \tag{15}$$

Instead of the conventional approach whereby Eq. (15) is used in Eqs. (6) or (7), Taylor series approximations to the strain and kinetic energies can be used to construct the approximate constraint.

$$\overline{U}_D = U_o + \sum_{i=1}^{n} u_i'(x_i - x_{oi}) \tag{16}$$

$$\overline{T}_D = T_o + \sum_{i=1}^{n} t_i'(x_i - x_{oi}) \tag{17}$$

In deriving Eqs. (16) and (17), the eigenvectors were assumed invariant with respect to changes in the design variables. The assumption implies that although the mode shape does not change in the region of the Taylor series, the frequency may vary. In fact, Miura and Schmit recommend this assumption as a means of reducing the computational burden of calculating the second derivative of an eigenvalue.[11] The assumption is also implicit in Venkayya's derivation of a scaling factor for frequency constraints.[15] The two approximations, Eqs. (16) and (17), are next combined to form a single approximate eigenvalue, referred to hereafter as the Rayleigh Quotient Approximation (RQA) in direct design variable space.

$$\bar{g}_{\mathrm{RQA}-D} = \pm\left(\frac{1}{\lambda^{\mathrm{allow}}}\frac{\overline{U}_D}{\overline{T}_D} - 1\right) \tag{18}$$

The same issue of an appropriate intermediate design variable is as pertinent for Eqs. (16) and (17) as for constructing a Taylor series directly for the eigenvalue. Should the strain and kinetic energies be approximated in direct or reciprocal design variable space? Recall that the only approximation thus far in deriving the RQA is that the eigenmode does not change. In essence, the initial eigenvector is an approximate mode shape for subsequent designs during optimization. Rayleigh originally observed that an approximate mode shape used in estimating the fundamental eigenvalue results in an estimate that is always higher than the true eigenvalue.[13] This implies that if the mode does change, Eq. (18) will overestimate the eigenvalue. However, approximating modal strain energy in reciprocal design variable space can alleviate the error.

$$\overline{U}_R = U_o + \sum_{i=1}^{n} = 1 u_i'(x_i - x_{o_i})\frac{x_{oi}}{x_i} \tag{19}$$

Equations (17) and (19) are combined to form the RQA using different design spaces for the modal energies, $\bar{\lambda}_{\mathrm{RQA}} = \overline{U}_R/\overline{T}_D$, expressed in constraint form as

$$\bar{g}_{\mathrm{RQA}} = \pm\left(\frac{\bar{\lambda}_{\mathrm{RQA}}}{\lambda^{\mathrm{allow}}} - 1\right) = \pm\left(\frac{\overline{U}_R}{\lambda^{\mathrm{allow}}\overline{T}_D} - 1\right) \tag{20}$$

Following Starnes and Haftka's criteria for hybrid constraints,[3] and observing that the u_i' terms in Eq. (16) are all positive, one concludes that approximating the modal strain energy in reciprocal design space yields a lower approximate eigenvalue. Whether this approximation is lower or higher than the actual eigenvalue is, in general, unknown. Nevertheless, as the first example to follow demonstrates, the approximate eigenvalue using Eq. (20) will be lower than the actual eigenvalue when the mode shape is (nearly) invariant. This situation is attractive for lower bound constraints for which Eq. (20) will be more conservative than Eq. (18). In contrast, Eq. (18) is always conservative for upper bound constraints when Eqs. (11) are exact and the mode shape may vary.

The derivative of the RQA constraint [Eq. (20)] is

$$\frac{\partial \bar{g}_{\mathrm{RQA}}}{\partial x_i} = \pm\frac{\left(\dfrac{x_{oi}}{x_i}\right)^2 u_i' - \bar{\lambda}_{\mathrm{RQA}} t_i'}{\lambda^{\mathrm{allow}}\overline{T}_D} \tag{21}$$

Notice that the sign of Eq. (21) can change as the design changes. If for a violated lower bound constraint the design variables with a positive derivative are reduced to increase the frequency, the positive derivative, initially dominated by the kinetic energy terms, may become negative as the strain energy term becomes more dominant. This behavior is consistent with intuition, which says that the frequency tends toward zero as the cross-sectional properties go to zero. This trait is not characteristic of a conventional TSA of the eigenvalue in direct or reciprocal design space – Eqs. (6) and (7), respectively (their gradients are constant in those spaces) – nor for Woo's GHC.

Special Case of High Nonstructural Mass

Structural designs with high nonstructural mass constitute a limiting case for the RQA. Venkayya and Tischler introduced modal mass ratios to characterize the degree of structural vs nonstructural mass.[10] If the mass matrix is considered as the sum of a structural mass matrix M_s and a constant (nonstructural) mass matrix M_c, then the modal mass ratios are defined as the nonstructural modal mass ratio

$$\gamma = \frac{\phi^t M_c \phi}{\phi^t M \phi} \tag{22}$$

and the structural modal mass ratio

$$\eta = \frac{\phi^t M_s \phi}{\phi^t M \phi} \tag{23}$$

By definition $\eta+\gamma=1$. In the limit, as nonstructural mass becomes dominant, $\gamma\rightarrow 1$ and $\eta\rightarrow 0$, the modal kinetic energy can be considered constant with respect to design changes, and the second term in the derivatives of the eigenvalue in Eqs. (15) and (21) can be neglected. In this case, the RQA reduces to a conventional TSA of the eigenvalue – either the reciprocal or direct variety, depending on which design space was used to approximate the modal strain energy. Starnes and Haftka's hybrid constraint reduces to the reciprocal TSA in this case, as well. The same reasoning for choosing the reciprocal design space for modal strain energy indicates that the reciprocal space would be more successful than the direct design space for a conventional TSA when optimizing structures with low structural modal mass ratios. A graphical illustration of this point is seen for the second example in the next section.

Computational Considerations

The only computational penalty for using RQA is that the optimizer has to deal with explicit nonlinear instead of linear constraints. The sensitivity analysis is the same except that two gradients must be stored for each frequency constraint instead of one. Additional "bookkeeping" is required to distinguish a frequency constraint from other types in order to apply the RQA. Otherwise, the method involves no more complexity than a conventional TSA of the eigenvalue.

Numerical Examples

Three-bar Truss

A simple three-bar truss (Fig. 1) is used to illustrate the differences among approximation techniques. A 10 lb (4.54 kg) point mass is at the free node. The fundamental frequency is constrained to be at least 1300 Hz. All three bars have an elastic modulus of 10 Mpsi (68.9 GPa), density of 0.1 lb/in.^{3} (2.8 g/cm^3), initial areas of 5.0 in.^{2} (32 cm^2), and minimum sizes of 0.001 in.^{2} (0.0065 cm^2). Effectively, no move limits were imposed, i.e., $f=10{,}000$ in Eq. (9). Conventional TSA's were made in both direct and reciprocal design spaces. For RQA results in Table 1, "direct" and "reciprocal" distinguish the design space used for approximating the strain energy. Direct RQA employs Eq. (18), whereas reciprocal RQA refers to Eq. (20). Subsequent examples employ Eq. (20) exclusively.

Due to symmetry, the two mode shapes for this system are always the same: one horizontal and one vertical. Since a constant mode shape was the only assumption made in deriving the RQA, Eq. (18) calculated the exact frequency and found the optimum in a single iteration. Because signs of the constraint's derivatives were not all the same, a conventional TSE in either space creates an infeasible design that is corrected in the next iteration. RQA using Eq. (20) was conservative, producing only feasible designs. The initial design had $\gamma=0.51$ and the final design, $\gamma=0.65$.

The design can be controlled by a single variable by recognizing two simplifications. One is that symmetry forces the two diagonal bars to have the same area. The other is that the vertical bar contributes no strain energy to the fundamental mode and will go to its minimum. Using Eq. (18), the optimum area of 3.736 in.^{2} (24.1 cm^2) for these two bars can be calculated by hand. The constraint functions are plotted in Fig. 2 as a function of the single variable controlling the two diagonal bars. The conservative nature of approximating strain energy in the reciprocal space is evident here. The reciprocal RQA can compensate for the error due to changes in the eigenvector; however, in this instance with an invariant mode shape it is overly conservative.

Cantilever Beam

The cantilever beam originally used by Turner,[16] shown in Fig. 3, was modeled using rod and shear panel elements. It is symmetric about the midplane and supports three nonstructural masses, each 30 lb (13.6 kg). Chord areas (A_1, A_2, A_3) and web thicknesses (t_1, t_2, t_3) were optimized for minimum weight, subject to a minimum fundamental frequency of 20 Hz. No other constraints were applied except minimum gages of $A_i=0.01$ in.^{2} (0.065 cm^2) and $t_i=0.001$ in. (0.0025 cm). Initial values were $A_i=1.0$ in.^{2} (6.45 cm^2) and $t_i=0.2$ in. (0.508 cm). Young's Modulus was 10.3 Mpsi (71 GPa), Poisson's ratio was 0.3, and the density was 0.1 lb/in.^{3} (2.8 g/cm^3).

Results are in Table 2 and Fig. 4. The final design was similar to those obtained by Turner,[16] Miura and Schmit,[11] and Woo[5]; however, the weight was slightly higher than for the latter two – entirely as a result of modeling and analysis differences. When Miura and Woo's final designs were analyzed in the small optimization program used for this paper, as well as in the larger Automated Structural Optimization System (ASTROS),[17] the frequency was 19.3 Hz. When the lower bound frequency was set to this value, the final designs were more nearly the same. Designs were feasible at every iteration using RQA without move limits ($f=10{,}000$) and the rate of convergence was faster than for Woo's results.

In order to examine the design space, the number of variables was reduced at a point near the optimum design. One design variable was linked to all the rod areas and one linked to the web thicknesses in the ratios given in Table 3. Taylor series expansions about this intermediate design point were used to plot constraint contours in Fig. 5. The failure reported by Miura and Woo for the direct TSA is evident in the poor quality of the approximating constraint surface to the actual highly nonlinear surface. In fact, since the direct TSA constitutes a linear programming problem, the optimizer always moves to a vertex in the design space, choosing to maximize the most effective variable while minimizing the rest. In the absence of severely restrictive move limits or other constraints to cut off the design space, a feasible design is never achieved.

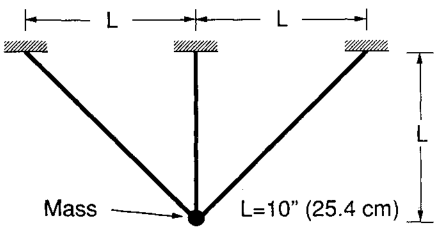

Fig. 1 Three-bar truss.

Table 1 Iteration history (weight) for three-bar truss

Iteration	Direct RQA, Eq. (18)	Reciprocal RQA, Eq. (20)	Direct TSA, Eq. (6)	Reciprocal TSA, Eq. (7)
0	19.14	19.14	19.14	19.14
1	10.57	11.50	10.36	0.015
2	10.57	10.67	10.56	141.3
3	--	10.57	10.56	24.02
4	--	10.57	--	13.30
5	--	--	--	10.87
6	--	--	--	10.57

Table 2 Cantilever beam final designs

	$A1$	$A2$	$A3$	$t1$	$t2$	$t3$	Weight	Frequency[a]
Turner[b]	0.91	0.485	0.14	0.037	0.034	0.023	7.27	19.8
Miura	0.871	0.441	0.108	0.044	0.041	0.026	7.00	19.3
Woo	0.866	0.442	0.109	0.046	0.041	0.025	7.01	19.3
RQA	0.875	0.466	0.129	0.035	0.031	0.020	6.92	19.3
RQA	0.955	0.484	0.140	0.038	0.034	0.022	7.44	20.0

[a]Frequencies calculated using CROD and CSHEAR elements (lumped mass) in ASTROS.
[b]Areas for Turner's design are the average for a linearly tapered rod.

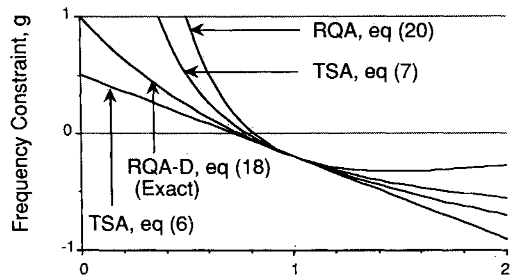

Fig. 2 Frequency constraint functions for three-bar truss.

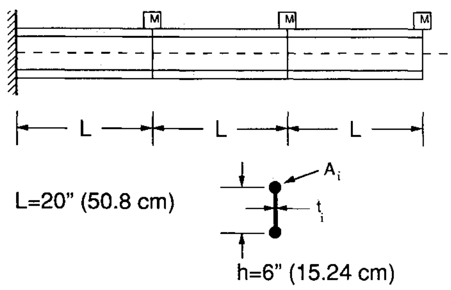

Fig. 3 Cantilever beam.

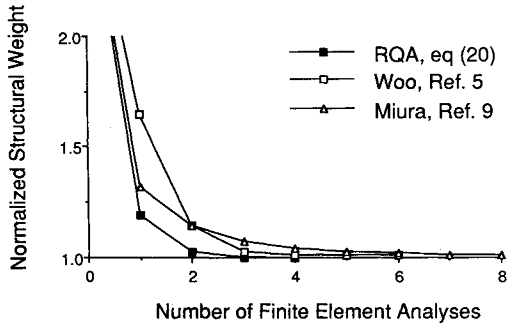

Fig. 4 Iteration history for cantilever beam.

Table 3 Cantilever beam intermediate design

$A1$	$A2$	$A3$	$t1$	$t2$	$t3$
1.0	0.56	0.125	0.10	0.08	0.06

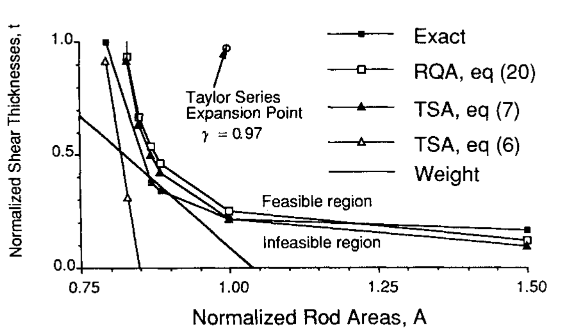

Fig. 5 Frequency constraint boundaries, $g(A,t)=0$, for beam.

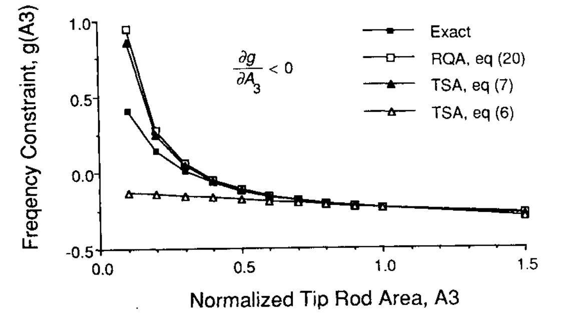

Fig. 6 Frequency constraint functions for beam intermediate design.

tutes a linear programming problem, the optimizer always moves to a vertex in the design space, choosing to maximize the most effective variable while minimizing the rest. In the absence of severely restrictive move limits or other constraints to cut off the design space, a feasible design is never achieved. Also, because the final nonstructural modal mass ratio is 0.98, the RQA closely follows the reciprocal TSA. Since the sign of both constraint derivatives is negative, Woo's GHC with $p=0$ and $n=1$ (equivalent to the hybrid constraint) would be identical to the Reciprocal TSA.

Consider next the constraint surfaces as a function of a single variable, the tip rod's area (A_3). Figure 6 shows a cut along the A_3 axis through the six-dimensional design space for the constraint functions at the intermediate design point ($\gamma=0.97$) of Table 3. It reflects the same comments mentioned above. Figure 7 shows the same functions constructed at the initial design point ($\gamma=0.88$), where the constraint derivative with respect to A_3 is positive instead of negative. Here the difference between the RQA and other approximations stands out. The RQA closely follows the actual constraint surface. Its derivative can change sign to match the curvature of the actual surface, whereas the TSA's derivative cannot change sign. In fact, the TSA's derivative is constant in the design space in which it was constructed. The advantage of the hybrid constraint (or Woo's GHC) is that, based on the constraint's sign, it chooses the more conservative surface, which is the direct TSA surface in this case. Neither TSA, however, represents well the actual constraint surface. Together, Figs. 6 and 7 illustrate the importance of a sign change in Eq. (21).

ACOSS

The Active Control of Space Structures (ACOSS) model II was developed by the Charles Stark Draper Laboratory.[18] The structure consists of two subsystems: the optical support structure and the equipment section. The two are connected by springs at three points to allow vibration isolation. In this problem, the equipment section at the base was disregarded and only the optical support structure, fixed at the three connection points, was considered. The finite element model for this modified ACOSS II (with supporting cross-members

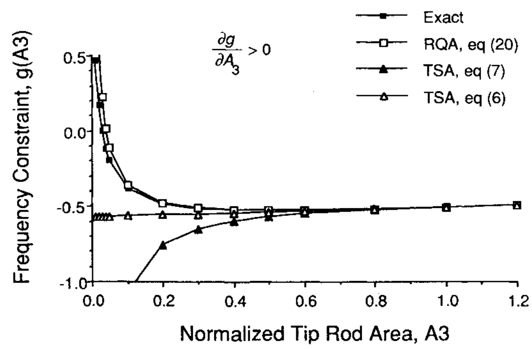

Fig. 7 Frequency constraint functions for beam initial design.

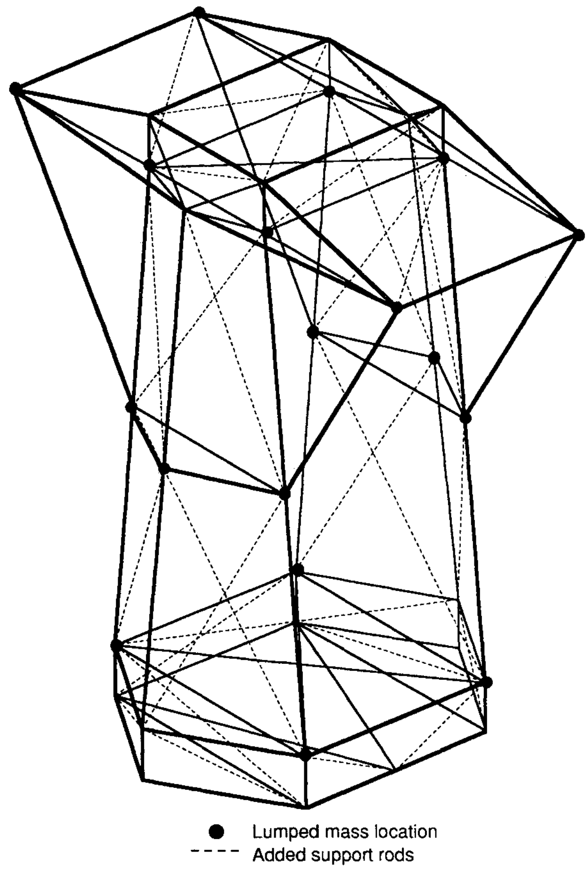

Fig. 8 Modified ACOSS II finite element model.

added), shown in Fig. 8, has 33 nodes (90 deg of freedom), 18 concentrated masses, and 113 rod elements made of graphite epoxy with Young's Modulus of 18.5 Mpsi (127 GPa), weight density of 0.055 lb/in.^{3} (1.5 g/cm^3) and initial areas of 10.0 in.^{2} (65 cm^2) for the truss members.

The structural weight was minimized using all 113 elements as design variables subject to a lower bound frequency of 2.0 Hz and minimum sizes of 0.1 in.^{2} (0.65 cm^2). The results in Fig. Fig. 9 show that RQA produced a significantly better final design – 10,000 lb (4,535 kg) – than TSA or the Optimality Criteria method.[11] A TSA of the eigenvalue in reciprocal space failed to converge to a feasible design even with $f = 1.5$. The TSA results in Fig. 9 are for $f = 1.5$ at iteration one, exponentially reduced at each iteration to $f = 1.2$. Still, the constraint was violated ($g > 0.1\%$) in the first 11 iterations and violated by more than 1% in the first 5 iterations. For RQA, the move limit factor ($f = 2$) prevented a feasible design until the second iteration, after which all subsequent designs were feasible. With no move limits ($f = 10{,}000$, not shown) RQA's first iteration and final design were feasible; however, some intermediate designs were violated by 1-4%. Initially, $\gamma = 0.42$ and at the

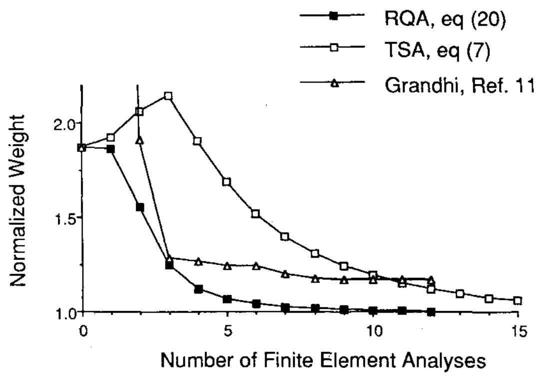

Fig. 9 Iteration history for modified ACOSS II.

final design $\gamma = 0.86$, showing why a reciprocal TSA eventually produced a feasible final design.

Conclusions

A new function for approximating frequency constraints during the solution of a structural optimization's approximate subproblem was developed. The motivation was to approximate a quantity more fundamental than the eigenvalue itself in order to improve the quality of the constraint approximation. Constructing approximations to the modal strain and kinetic energies independently resulted in more accurate constraint evaluation without any additional computational burden. Use of the reciprocal design space for the modal strain energy approximation tended to compensate for the error in assuming an approximate mode shape during optimization. The resulting Rayleigh Quotient Approximation has the important and unique characteristic that its derivative can change sign to more closely follow the actual constraint surface. Although the numerical examples were simple, the theory itself does not share their limitations. Hence, future work should examine multiple frequency constraints and space frame applications. Nonetheless, the examples presented did demonstrate that the RQA is more accurate than other approximations of the fundamental eigenvalue and permits stable convergence with generous move limits.

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References

[1]Schmit, L. A., and Farshi, B., "Some Approximation Concepts for Structural Synthesis," *AIAA Journal*, Vol. 2, No. 5, 1974, pp. 692-699.

[2]Schmit, L. A., and Miura, H., "Approximation Concepts for Efficient Structural Synthesis," NASA CR-2552, 1976.

[3]Starnes, J. H., and Haftka, R. T., "Preliminary Design of Composite Wings for Buckling, Strength, and Displacement Constraints," *Journal of Aircraft*, Vol. 16, No. 8, 1979, pp. 564-570.

[4]Fleury, C., and Braibant, V., "Structural Optimization – A New Dual Method Using Mixed Variables," *International Journal for Numerical Methods in Engineering*, Vol. 23, No. 3, 1986, pp. 409-428.

[5]Woo, T. H., "Space Frame Optimization Subject to Frequency Constraints," *Proceedings of 27th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference*, San Antonia, TX, May 19-21, 1986, pp. 103-115.

[6]Fleury, C., "Efficient Approximation Concepts Using Second Order Information," *Proceedings of 29th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference*, Williamsburg, VA, April 18-20, 1988, pp. 1685-1695.

[7]Vanderplaats, G. N., and Salajegheh, E., "An Efficient Approximation Technique for Frequency Constraints in Frame Optimization," *International Journal for Numerical Methods in Engineering*,

Vol. 26, No. 5, 1988, pp. 1057-1069.

[8]Mills-Curran, W. C., and Schmit, L. A., "Structural Optimization with Dynamic Behavior Constraints," *AIAA Journal*, Vol. 23, No. 1, 1985, pp. 132-138.

[9]Vanderplaats, G. N., and Salajegheh, E., "A New Approximation Method for Stress Constraints in Structural Synthesis," *AIAA Journal*, Vol. 27, No. 3, 1989, pp. 352-358.

[10]Venkayya, V. B., and Tischler, V. A., "Optimization of Structures with Frequency Constraints," *Computational Methods Nonlinear Solids and Structural Mechanics,* American Society of Mechanical Engineers, AMD-54, 1983, pp. 239259.

[11]Miura, H., and Schmit, L. A., "Second Order Approximation of Natural Frequency Constraints in Structural Synthesis," *International Journal of Numerical Methods in Engineering,* Vol. 13, No. 2, 1978, pp. 337-351.

[12]Grandhi, R. V., and Venkayya, V. B., "Structural Optimization with Frequency Constraints," *28th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference,* Monterey, CA, April 6-8, 1987, pp. 32-329.

[13]Rayleigh, J. W. S., *The Theory of Sound*, 2nd ed., Vol. 1, Dover Publications, NY, 1945, pp. 91-129.

[14]Fox, R. L., and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426-2429.

[15]Canfield, R. A., Grandhi, R. V., and Venkayya, V. B., "Structural Optimization with Stiffness and Frequency Constraints," *Mechanics of Structures and Machines*, Vol. 17, No. 1, 1989.

[16]Turner, M. J., "Design of Minimum Mass Structures with Specified Natural Frequencies," *AIAA Journal*, Vol. 5, March 1967, pp. 406-412.

[17]Neill, D. J., Johnson, E. H., and Canfield, R. A., "ASTROS-A Multidisciplinary Automated Structural Design Tool," *Proceedings of 28th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference,* AIAA, New York, 1987, pp. 4453; also, *Journal of Aircraft* (to be published).

[18]Handerson, T., "Active Control of Space Structures (ACoSS) Model 2," Rept. C-5437, Charles Stark Draper Laboratory, Inc., Cambridge, MA, Sept. 1981.